A geometric approach to the renormalization of complex networks
Networks: a change of paradigm

Graph representations of discrete complex systems

Networks are changing the way in which we study and understand real complex systems
Real networks talk a common language

- A large number of interacting elements
- Globally connected (percolated)
- Self-organization, lack of centralized control
- Between order and disorder
- Low average number of connections
- Communities
- Hierarchies
- Multilayer
- Temporality
- …
Real networks talk a common language

- Small-world property
- Clustering
- Heterogeneity
Real networks talk a common language

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\[ c_i = \frac{2T_i}{k_i(k_i - 1)} \]

- C. Elegans cell 0.34
- Cat1 area 0.66
- Drosophila1 cell 0.24
- Macaque1 area 0.77
- Mouse1 area 0.45
- Human6 area 0.61
- Rat1 area 0.89
Real networks talk a common language

- Small-world property
- Clustering
- Heterogeneity

Scale-free degree distributions of connectivity in networks with characteristic exponent $2<\gamma<3$

homogeneous functions $\rightarrow$ scale invariance
exact form of self-similarity

$$P(\lambda k) = \lambda^{-\gamma} P(k)$$
Big questions I: Symmetries

- **core concept in physics**: dictate the structure of matter, the forces in nature, and the laws of physics; at the root of all our **modern theories**

- **invariance** to transformations and underlying **conservation laws**

- restrict how a physical system can be or behave, so that they **force** it to create **recognizable patterns**

- this process works both ways: symmetry restricts output; the underlying **symmetry can be inferred from patterns**

Can we find symmetries in complex networks?
Distances = shortest paths
Box covering method
Fractality (like in fractal geometry) and topological renormalization

Topological distances present a serious problem to distinguish close from far away: because of the SMALL WORLD(or ultra small world) property, not a wide range of length scales

\[ l(N) \sim \log N \]

SMALL WORLD property:
the diameter or maximum distance grows slower than any polynomial

Complex networks and hidden metric spaces

- Underlying geometries provide a simple explanation for the observed structure of real networks
- Real networks can be successfully embedded into hidden metric spaces

The clue for the connection between complex networks and metric spaces is clustering

Clustering is a by product of the triangle inequality in the hidden metric space

\[ AC \leq AB + BC \]

Hidden metric spaces vs explicit spaces

+ universal probability of connection depending on distances

...distances in latent space incorporate ALL factors that contribute to connectivity,

distance in explicit spaces is typically one factor among many...

The connection probability $p$ is an integrable function of the effective distance $d$. A. Serrano, D. Krioukov, M. Boguñá, Phys. Rev. Lett. 100, 078701 (2008)

$k \rightarrow \text{popularity, mass, importance}$

d $\rightarrow \text{similarity}$
The S1 model

- $\rho(\kappa)$ controls the degree distribution, SF $\rho(\kappa) \approx P(k)$
- independently, $\beta$ controls the level of clustering, strong clustering
- given $\beta$, parameter $\mu = \left(\frac{\gamma - 2}{\gamma - 1}\right)^2 \frac{(\beta - 1) < k >}{2\delta k_0^2}$ controls the average degree

1 < $\beta$ < 2 or 2 < $\gamma$ < 3

small-world!!! but underlying metric!

The S1 model - specifications

the S1 model is
the only entropy maximizing ensemble
that can produce sparse heterogeneous networks with
the small world property and non-zero clustering

Statistical mechanics approach:

We take advantage of the analogy between the model
ensemble and a system of fermions where particles
correspond to links that can be in different states with
energy depending on geometric distance

Grand-canonical statistical ensemble of random graphs
Chemical potential related to $\mu \rightarrow <L>$
Temperature $(1/\beta) \rightarrow <E>$; $E f(d_a, \kappa)$'

$\epsilon_{ij}=f(x_{ij})$

proof to be published soon

S1 isomorphic to H2 in hyperbolic space

S1: $p$ is a function of distance on the circle + hidden degrees

$P_{ij} = \frac{1}{1 + \left( \frac{d_{a,ij}}{\mu \kappa_i \kappa_j} \right)^\beta}$

H2: $p$ is only a function of distance

$r = R - \frac{2}{\zeta} \ln \left[ \frac{\kappa}{\kappa_0} \right]$  

$p_{ij} = \frac{1}{1 + e^{\frac{\beta}{2}(x_{ij}-R)}}$


S1 is isomorphic to model H2 in hyperbolic space

S1: \( p \) is a function of distance on the circle and hidden degrees

\[
p_{ij} = \frac{1}{1 + \left( \frac{d_{a,ij}}{\mu \kappa_i \kappa_j} \right)^\beta}
\]

Newtonian, gravity model

H2: \( p \) is only a function of distance

\[
p_{ij} = \frac{1}{1 + e^{\frac{\beta}{2} (x_{ij} - R)}}
\]

Einsteinian, latent space

relation between hidden degree and radius

\[
r = R - 2 \ln \left[ \frac{\kappa}{\kappa_0} \right]
\]


The S1 model as a topology generator

- Power-law degree distributions, High levels of clustering, degree correlations, Small-world property

The model explains a non-obvious symmetry in real networks, the self-similarity of the nested hierarchy of subgraphs obtained by degree-thresholding

Many real scale-free networks are self-similar \((P(k), k_{\text{nn}}(k), c(k))\) with respect to a simple degree-thresholding renormalization procedure (purely topological!!)

The \(S_1\) class of hidden variable models with underlying metric spaces are able to accurately reproduce the observed topology and self-similarity properties.

**Degree thresholding**

\[ k > k_T \]
\[ k_i \rightarrow k_i/\langle k(k_T) \rangle \]

**Model**

Thresholding of the hidden variable \(K\)

The ensemble is self-similar with respect to \(T\) when any subgraph belong to the original ensemble

\[ G_T(\{\alpha\}) = G(\{\alpha_T\}) \]

Real networks

Model

The percolation threshold (and for any phase transition whose critical points depend monotonously on the average degree) is zero in self-similar networks with subgraphs of growing average degree, without the need of the locally tree-like assumption (so it is valid for networks with clustering) proof by contradiction.

M. A. Serrano, D. Krioukov, M. Boguñá
PRL 100, 078701 (2008)
PRL 106, 048701 (2011)
The S1/H2 model and real networks

The S^2 / H^2 model can be used to embed real networks

Coordinates of nodes
hidden degree and angular position on the similarity space
that maximize the congruency between the network and the model
maximize the likelihood that the observed topology has been
produced by the model

\[ L(p_{mr}) = \prod_{m,r} p_{mr}^{a_{mr}} (1 - p_{mr})^{1-a_{mr}} \]

M. Boguñá, F. Papadopoulos, D. Krioukov
Nature Communications 1, 62 (2010)
Communities based on geometry

Natural Communities

densely populated zones separated by void angular sectors

Critical gap method CGM

nodes in the same community have an angular separation smaller than a critical value (selected to have maximum congruency with topological communities in the backbone)


http://morfeo.ffn.ub.edu/wta1870-2013/
Navigability

most travelers reach destination; paths follow closely topological shortest paths

Hyperbolic distances can be used to route information

GREEDY ROUTING

Each node knows its own location, the location of its neighbors, and that of the destination.

The message can be routed without global knowledge of the network topology.

**Success rate**: number of messages that reach destination

**Stretch**: length of the paths relative to the corresponding shortest paths

BIG questions II: the problem of scales

- **Multiple scales** (of length, time…) **coexist** in real systems: critical points and phase transitions, crashes and avalanches, polymers and macromolecules, correlated electronic systems, for quantum field theory, deep learning, **complex networks**?

- **zooming in** and **out**

- How does a system behave at different scales and how do the different scales **interplay**?
Renormalization group (RG)
A conceptual framework in theoretical physics for the investigation of the changes of a physical system as viewed at different length scales

In Statistical Physics
the RG explains the universality of critical behavior in phase transitions (fluids, ferromagnets, liquid mixtures, alloys...) at the critical point of a phase transition --Ising model, percolation, ...--, fluctuations become scale-invariant and occur at all length scales

Block spin model, L. P. Kadanoff
the modern development of the RG began with Leo’s intuitive construction of “block spins”, in which he made explicit the idea of averaging over short-range fluctuations
Geometric RG for complex networks

Start from the map of a complex network embedded in the $S_1$ hidden metric space

Recursive averaging over short range interactions

- Coarse-graining + rescaling applied iteratively, this defines a RG flow
- Impose that the connection probability maintains its form
- Look at the behavior of parameters and observables in the flow

The transformation can be iterated $O(\log N)$ in real networks

The transformation has the semi-group structure,

$$r_n \text{ steps} = (r_{1\text{step}})^n$$

Real SF networks are self-similar

**Internet at the AS level**
N=23748
\( \gamma=2.17 \)  \( \beta=1.44 \)
\( <k>=4.92 \)
\( <c>=0.61 \)

**Metabolic network**
N=1436
\( \gamma=2.6 \)  \( \beta=1.3 \)
\( <k>=6.57 \)
\( <c>=0.54 \)

**Music**
N=2476
\( \gamma=2.7 \)  \( \beta=1.1 \)
\( <k>=16.66 \)
\( <c>=0.82 \)

**World airports Openflights**
N=3397
\( \gamma=1.88 \)  \( \beta=1.7 \)
\( <k>=11.32 \)
\( <c>=0.63 \)

**Human protein-protein interaction network**
N=4100
\( \gamma=2.25 \)  \( \beta=1.0 \)
\( <k>=6.52 \)
\( <c>=0.09 \)

**Word adjacency in “On the Origin of the Species”**
N=7377
\( \gamma=2.25 \)  \( \beta=1.0 \)
\( <k>=11.99 \)
\( <c>=0.47 \)
The community structure is preserved in the RGN flow.

- High-modularity partitions of the original network from much smaller versions.
- This is a consequence of the fact that angular coordinates of nodes in the circle encode the community structure of the network and that the RGN preserves the angular ordering of nodes.
- A new and efficient multiscale community detection algorithm.
Real SF networks are self-similar

\[ P(x_{ij}^{(l)}) \]

\[ l = 0 \quad l = 1 \quad l = 2 \quad l = 3 \quad l = 4 \quad l = 5 \]

\[ P(x_{ij}^{(l)}) \]

\[ l = 0 \quad l = 1 \quad l = 2 \]

\[ P(x_{ij}^{(l)}) \]

\[ l = 0 \quad l = 1 \quad l = 2 \]

\[ P(x_{ij}^{(l)}) \]

\[ l = 0 \quad l = 1 \quad l = 2 \]

\[ P(x_{ij}^{(l)}) \]

\[ l = 0 \quad l = 1 \quad l = 2 \quad l = 3 \]

Empirical connection probabilities

\[ p_e = \frac{1}{1 + \left( \frac{R \Delta \theta_e}{\mu(\kappa_m \kappa_n)e} \right)^\beta} \]

\[ x_{ij}^{(l)} = R^{(l)} d_{a,ij}^{(l)} / (\mu^{(l)} \kappa_i^{(l)} \kappa_j^{(l)}) \]
The S\(_1\) model IS RENORMALIZABLE:
RGN realizations at any scale belong to the same ensemble except for the fact that the average degree of the renormalized network changes

We can prove analytically that
the connection probability maintains its original form and
the model preserves the semigroup structure if

\[
K^{(l)}_i = \left( \sum_{j=1}^{r} (K^{(l-1)}_j)^\beta \right)^{1/\beta}
\]

Hidden degree of supernode \(i\) in
layer \(l\):
\(\beta\)-norm of the hidden degrees of its
associated nodes \(j\)

\[
\theta^{(l)}_i = \left( \frac{\sum_{j=1}^{r} \left( \theta^{(l-1)}_j K^{(l-1)}_j \right)^\beta}{\sum_{j=1}^{r} (K^{(l-1)}_j)^\beta} \right)^{1/\beta}
\]

Angular coordinate of supernode \(i\) in
layer \(l\):
\(\beta\)-norm of the angular positions of its
associated nodes \(j\) weighted by the
corresponding hidden degrees

\[
p_e = \frac{1}{1 + \left( \frac{R \Delta \theta_e}{\mu(K_m K_n)_{e}} \right)^\beta}
\]

\[
p'_{ij} = 1 - \prod_{e=1}^{r^-2} (1 - p_e)
\]

\(N\) large
Angular distance inside blocks
smaller than the average

\(\beta\) and \(\gamma\) do not change!

If the original distribution of
hidden degrees is a power law,
exponent \(\gamma\), the hidden degree
distribution in the renormalized
layers is also asymptotically a
power law with the same
exponent, whenever \(\gamma < 3\) or
\((\gamma - 1)/2 < \beta\).
The average degree changes in the RGN flow

\[ \langle k \rangle^{(l)} = r^\nu \langle k \rangle^{(l-1)} \]

The flow is \( \gamma \)-dominated (\( \beta > \gamma - 1 \))

\[ \nu = \frac{2}{\gamma - 1} - 1 \]

The flow is \( \beta \)-dominated (\( \beta < \gamma - 1 \))

\[ \nu = \frac{2}{\beta} - 1 \]

**Phase I:** \( \nu > 0 \) and the network flows towards a fully connected graph; the RGN flow progressively selects more and more long range connections

**Phase II:** \( \nu < 0 \) and the network flows towards a one-dimensional ring

**Red thick line:** \( \nu = 0 \), transition between small-world and non-small-world

**In region III,** the degree distribution loses its scale-freeness along the flow

For most real networks, the average degree is a relevant observable

Connectivity phase diagram
For each link between two nodes $i$ and $j$ present in the renormalized network

i) Compute
\[ \mu_{new} = \mu_{target} \frac{\langle k_{new} \rangle}{\langle k_{target} \rangle} \]

ii) Compute the probability of connection according to $\mu_{new}$
\[ p_e = \frac{1}{1 + \left( \frac{R \Delta \theta_e}{\mu(\kappa_m \kappa_n)\epsilon} \right)^{\beta}} \]
\[ q_{ij} = \frac{p_{ij}(\mu_{new})}{p_{ij}(\mu)} \]

so that the probability of existence of a link in the pruned version is
\[ p_{ij}(\mu_{new}) \]

iii) Readjust and iterate
Order–disorder transitions
Control parameter measured at stationary state over 100 realizations

**ISING MODEL:** Nodes have spins values $s=+1$ or $-1$. Minimization of energy $H = - \sum_{i<j} s_i s_j$
Initial condition: $s=1$ for all nodes
Metropolis-Hastings algorithm: a spin is picked up at random and flipped if the resulting configuration has lower or equal energy, otherwise it is flipped with a probability $e^{-\Delta H/T}$

$T$ is the control parameter and the order parameter is the magnetization $m = \frac{1}{N} \sum_i s_i$

**SIS MODEL:** Nodes are infected +1 or susceptible 0. Infection and recovery are Poisson processes.
Initial condition: all nodes infected
Continuous-time Gillespie algorithm: an event (infection or recovery) is selected according to its associated probability and the state of the system and the probabilities are updated

The infection rate $\lambda$ is the control param and the prevalence $\rho(t) = \frac{1}{N} \sum_i n_i(t)$ is the order param

**KURAMOTO MODEL:** Nodes have a natural frequency and a phase and oscillators interact

$\dot{\theta}_i = \omega_i + \sigma \sum_{j \neq i} a_{ij} \sin(\theta_j(t) - \theta_i(t))$

Initial condition: phases and frequencies are taken from uniform distributions $U(-\pi, \pi)$ and $U(-0.5, 0.5)$
Heun’s method: numerical integration of the system of ODEs

The control parameter is the coupling strength $\sigma$ and the order parameter is $r(t) = \frac{1}{N} \left| \sum_i e^{i\theta_i(t)} \right|$
In large-scale simulations, downscaled network replicas could serve as an alternative or guidance to sampling methods, or for fast-track exploration of rough parameter spaces in the search of relevant regions.

Downscaled versions of real networks could also be applied to perform finite size scaling, which would allow for the determination of critical exponents from single snapshots of their topology.
Multiscale navigation is different from greedy routing in a single layer, exploits information at different scales.

The process takes place in the real network but the path is decided according to greedy routing between supernodes in upper layers.

Slightly different coarse-graining requiring nodes connected inside supernodes
• the probability of connection is preserved
• self-similarity of the topology is effectively preserved
Multiscale navigation

Success rate: fraction of successful paths

Stretch: ratio topological length of successful paths vs corresponding shortest path
In summary

Models of complex networks based on hidden metric spaces currently offer the most powerful explanation for their observed structure.

RENORMALIZATION GROUP FOR COMPLEX NETWORKS

- explains symmetries observed in real networks
- offers a basis for a new approach to explore critical phenomena and universality
- affords us a multiscale unfolding and immediate practical applications: multiscale navigation, downscaled network replicas
What is next?

MERCATOR
Command-line tool for producing maps of complex networks in the hyperbolic plane

REVERSING THE RG

NETWORK DIMENSIONALITY

GEOMETRY OF BRAIN NETWORKS

Septentrionalium Terrarum descriptio
The first map dedicated to the North Pole Mercator 1595

arXiv.org > q-bio > arXiv:1801.06079
Quantitative Biology > Neurons and Cognition
Navigable maps of structural brain networks across species
Antoine Allard, M. Ángeles Serrano
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