Understanding transition to turbulence in fluids: computing hidden flows and beyond

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An appetizer:

DNS (Direct Numerical Simulation of Navier-Stokes flows)

WVF
Motivation: understanding transition in fluids

"Yet not every solution of the equations of motion, even if it is exact, can actually occur in Nature. Those which do must not only obey the equations of fluid dynamics, but also be stable."

Landau & Lifshitz *Fluid Mechanics* (1959)
Introduction: history of hydrodynamic stability (I)

Thermal convection flows:
- Lord Rayleigh (1880s):

Centrifugal flows:
- G.I. Taylor (1920s)
**Introduction: history of hydrodynamic stability (II)**

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**Landau-Hopf scenario (1940s)**

L. Landau: 

E. Hopf:
Route to turbulence: Ruelle-Takens-Newhouse

Flow regimes

WVF  WVF

MWVF  WVF

WVF  WVF

Frequency spectra
Route to turbulence via (supercritical) bifurcations:

\[ \frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{v} \]

\[ \nabla \cdot \mathbf{v} = 0 \]

Dynamical System

\[ \dot{x} = f(x, \text{Re}) \]

Linear stability of equilibria

\[ f(x_0, \text{Re}) = 0, \quad p(\lambda) = \det(Df - \lambda I) \]
Centrifugal-thermal transition scenarios:

- For some $Re$, basic flows become linearly unstable (local bifurcation)
- Laminar secondary flows (steady, time periodic or almost-periodic)
- Experimental evidence of RTN route to chaos (turbulence)
- Other scenarios may be at work (period doubling, intermittency)
- Bifurcation cascade via local linear instabilities of secondary flows
- Turbulence: low dimensional strange attractor, or:
  
  turbulence is a stable regime

... but ... what about shear flows?
The Reynolds problem: pipe flow (1880s)

O. Reynolds

Main discrepancies with classical transition scenarios:

- Basic flow is **always** linearly stable (no local bifurcation)
- No laminar secondary flows have ever been observed
- Turbulence directly appears for $\text{Re} > 2000$ (turbulent spots: puffs)
The three canonical shear flows:

- **Plane Couette Flow**

- **Plane Poiseuille Flow**

- **Pipe Flow**

- **Linearly stable** $\forall \text{Re}$ (proof)

- **Turbulent for** $\text{Re} \sim O(10^2)$

- **Linearly unstable for** $\text{Re} = 5772$

- **Turbulent for** $\text{Re} \gtrsim 1000$

- **Linearly stable** $\forall \text{Re} \leq 10^7$

- **Turbulent for** $\text{Re} \sim O(10^3)$
The double threshold problem

- Are (unstable) laminar solutions responsible for the critical threshold?
- If yes, how can we find them?
A model with narrow basin of attraction

\[ \dot{x} = -\frac{1}{Re} x + y - y \sqrt{x^2 + y^2} \]

\[ \dot{y} = -\frac{2}{Re} y + x \sqrt{x^2 + y^2} \]

- **Steady solution:** \((x_0, y_0) = (0, 0)\)
- **Linearly stable** \(\forall Re > 0:\)
  \[ J = \begin{pmatrix} -Re^{-1} & 1 \\ 0 & -2Re^{-1} \end{pmatrix} \]
  \[ \lambda_1 = -\frac{1}{Re} \]
  \[ \lambda_2 = -\frac{2}{Re} \]
**Pipe flow: mathematical formulation**

**Hagen-Poiseuille flow:**

\[
\mathbf{u}_b = u_b \hat{r} + v_b \hat{\theta} + w_b \hat{z} = (1-r^2) \hat{z}
\]

\[
\text{Re} = \frac{a U_{cl}}{\nu}
\]

\[
U_{cl} = -\frac{\Pi_0 a^2}{4\rho \nu}
\]

**Computational domain:** \((r, \theta, z) \in \mathcal{D} = [0, 1] \times [0, 2\pi] \times [0, \Lambda]\)

**Length (radii units):** \(\Lambda_{\text{short}} = 6.4\pi \sim 20, \quad \Lambda_{\text{long}} = 32\pi \sim 100\)

**Navier-Stokes equation for the perturbation fields:**

\[
\mathbf{v} = \mathbf{u}_b + \mathbf{u}, \quad p = p_b + q
\]

\[
\partial_t \mathbf{u} = -\nabla q + \frac{1}{\text{Re}} \Delta \mathbf{u} - (\mathbf{u}_b \cdot \nabla) \mathbf{u} - (\mathbf{u} \cdot \nabla) \mathbf{u}_b - (\mathbf{u} \cdot \nabla) \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0
\]

\[
\mathbf{u}(r, \theta, z + \Lambda, t) = \mathbf{u}(r, \theta, z, t), \quad \mathbf{u}_{\text{wall}} = 0
\]
Pipe flow: spectral method & time stepper

Spectral solenoidal approximation of velocity field:

\[
\mathbf{u}(r, \theta, z, t) = \sum_{l,n,m} a_{lnm}(t) \Phi_{lnm}(r, \theta, z), \quad \nabla \cdot \Phi_{lnm} = 0, \quad \Phi_{lnm}(r = 1) = 0
\]

Dynamical System for the \(a\)-coefficients:

\[
\begin{align*}
A \frac{da}{dt} &= B a + b(a, a).
\end{align*}
\]

Initial perturbation:

\[
\begin{align*}
\mathbf{u}_0^{2D} (n = 1) &= C^{2D} e^{i\theta} \mathbf{v}_1(r) \\
\mathbf{u}_0^{3D} (n = 0, \pm 1, \ k_o l \in [1.5, 2.2]) &= \sum_{l,n} C^{3D}_{ln} e^{i(n \theta + k_o l z)} \mathbf{v}_n(r) + \text{c.c.}
\end{align*}
\]
Edge states: short pipe ($\Lambda = 10$)

Energy-Amplitude:

$$\varepsilon(u) = \frac{3}{\pi\Lambda} \int_{\mathcal{D}} u^\dagger \cdot u \, d\mathcal{D}. $$

$$A(u) = \sqrt{\varepsilon(u)}$$

Tracking method:

- 1. Scaling $\uparrow\downarrow \varepsilon(u_0)$
- 2. Bisection shooting
- 3. Critical trajectory:

A typical exploration:

![Graph showing turbulent and laminar runs with time evolution.](image-url)
Newton-Krylov solver for travelling waves

Initial seed (edge state):

\[ a^{(0)} = \]

Continuation:

Asymmetric TW:

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Symmetry breaking pitchfork bifurcation

Symmetric TW:

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Re = 2875, \( \Lambda = 10 \)
Edge states: long pipe ($\Lambda = 100$)

Critical:

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Localized turbulence (puff):

Newton-Krylov:

WVF
Subcritical Taylor-Couette flow: intermittency

Turbulent spots: Edge state:

Newton-Krylov:

WVF
Conclusions

• Identifying subcritical transition mechanisms needs top-notch numerics
• DNS computations (time-stepping) are limited in scope
• Unstable (hidden) flows may destabilize base laminar flows
• Identifying these hidden flows requires new techniques:
  Edge state tracking, Newton-Krylov continuation, etc...

References